

Non-Gaussian corrections to the probability distribution of the curvature perturbation from inflation

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Abstract. We show how to obtain the probability density function for the amplitude of the curvature perturbation, \mathcal{R} , produced during an epoch of slow-roll, single-field inflation, working directly from n -point correlation functions of \mathcal{R} . These n -point functions are the usual output of quantum field theory calculations, and as a result we bypass approximate statistical arguments based on the central limit theorem. Our method can be extended to deal with arbitrary forms of non-Gaussianity, appearing at any order in the n -point hierarchy. We compute the probability density for the total smoothed perturbation within a Hubble volume, ϵ , and for the spectrum of ϵ . When only the two-point function is retained, exact Gaussian statistics are recovered. When the three-point function is taken into account, we compute explicitly the leading slow-roll correction to the Gaussian result.

Keywords: Inflation, Cosmological perturbation theory, Physics of the early universe

1. Introduction

There is now a good deal of observational evidence that the generic predictions of the inflationary scenario are realised in the spectrum of density perturbations in our universe [1, 2, 3, 4]. For slow-roll inflation driven by a scalar ϕ , these predictions are:

- (i) a nearly scale-invariant spectrum of fluctuations on all scales accessible to cosmological observation;
- (ii) for inflation near the theoretically motivated energy scale $M_{\text{GUT}} \simeq 10^{16}$ GeV, these fluctuations should have magnitude $\delta\rho/\rho \simeq 10^{-5}$; and
- (iii) the fluctuation spectrum should exhibit Gaussian statistics, in the sense that the probability distribution of the density fluctuation should be approximately normally distributed.

(For a review of the inflationary paradigm and its predictions, see *eg.*, Refs. [5, 6].) Predictions (i) and (ii) can be obtained using the standard techniques of quantum field theory. This calculation is now classical [7, 5, 6] and relies only on the fact that the vacuum fluctuation of a scalar field in de Sitter space with Hubble parameter H is roughly $\delta\phi = H/2\pi$ [8, 9, 10, 11]. The third prediction—that the fluctuation spectrum is Gaussian—is less transparent. It follows from the fact that the inflaton perturbation, which is commonly expressed in terms of the so-called comoving curvature perturbation[‡] [13] \mathcal{R} , is treated as a free field,

$$\mathcal{R}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \mathcal{R}(t, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (2)$$

where there is no coupling between the $\mathcal{R}(t, \mathbf{k})$ for different \mathbf{k} . With this understanding, Eq. (2) means that \mathcal{R} does not interact either with itself, or any other particle species in the universe. The real-space field $\mathcal{R}(t, \mathbf{x})$ is obtained by summing an infinite number of independent, identically distributed, uncorrelated oscillators. Under these circumstances the Gaussianity of $\mathcal{R}(t, \mathbf{x})$ follows from the central limit theorem [14], given reasonable assumptions about the individual distributions of the $\mathcal{R}(t, \mathbf{k})$. The exact form of the distributions of the $\mathcal{R}(t, \mathbf{k})$ is mostly irrelevant when making statements about the inflationary density perturbation.

[‡] $\mathcal{R}(t, \mathbf{x})$ expresses the relative expansion of a given local neighbourhood of the universe with respect to the background, in a gauge where observers in free-fall with the expansion see no net momentum flux. This is the so-called comoving gauge. In particular, during inflation the scalar field fluctuation is zero on comoving slices. The metric in this gauge is thus

$$ds^2 = -N^2(t, \mathbf{x}) dt^2 + a^2(t) e^{2\mathcal{R}(t, \mathbf{x})} \gamma_{ij} (dx^i + N^i(t, \mathbf{x}) dt)(dx^j + N^j(t, \mathbf{x}) dt), \quad (1)$$

where γ_{ij} is the metric on unperturbed spatial slices, which we take to be flat and is subject to the condition $\det \gamma = 1$; and $a(t)$ is the unperturbed scale factor of the universe. The functions N and N^i are the so-called lapse function and shift vector, and are determined by algebraic constraint equations in terms of the matter content and the metric fields (\mathcal{R} , a and γ), once the gauge is fixed. Our sign convention for the metric is “mostly plus”, $(-, +, +, +)$. The advantage of working with the perturbation $\mathcal{R}(t, \mathbf{x})$ is that it mixes scalar fluctuations from the metric and matter sectors in a gauge invariant way [12].

In conventional quantum field theory, all details of \mathcal{R} and its interactions are encoded in the n -point correlation functions of \mathcal{R} (or their Fourier transforms), written $\langle \text{out} | \mathcal{R}(t_1, \mathbf{x}_1) \cdots \mathcal{R}(t_n, \mathbf{x}_n) | \text{in} \rangle$. Working in the Heisenberg picture, where the fields carry time dependence but the states $\{| \text{in} \rangle, | \text{out} \rangle\}$ do not, these functions express the amplitude for the early-time vacuum $| \text{in} \rangle$ to evolve into the late-time vacuum $| \text{out} \rangle$ in the presence of the fields $\mathcal{R}(t_i, \mathbf{x}_i)$. Given the n -point functions for all n at arbitrary \mathbf{x} and t , one can determine $\mathcal{R}(t, \mathbf{x})$ [15], at least in scattering theory. In the context of the inflationary density perturbation, these vacuum evolution amplitudes are not directly relevant. Instead, one is interested in the equal time expectation values $\langle \text{in} | \mathcal{R}(t, \mathbf{x}_1) \cdots \mathcal{R}(t, \mathbf{x}_n) | \text{in} \rangle$, which can be used to measure gravitational particle creation out of the time-independent early vacuum $| \text{in} \rangle$ during inflation. These expectation values are calculated using the so-called “closed time path formalism,” which was introduced by Schwinger [16] (see also [17, 18, 19, 20]). In this formalism there is a doubling of degrees of freedom, which is also manifest in finite temperature calculations [21, 22]. This method has recently been used [23, 24, 25] to extend the computation of the correlation functions of \mathcal{R} beyond tree-level.

Knowledge of the expectation values of \mathcal{R} in the state $| \text{in} \rangle$ is sufficient to predict a large number of cosmological observables, including the power spectrum of the density perturbation generated during inflation [9, 10], and the two- and three-point functions of the cosmic microwave background (CMB) [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36] temperature anisotropy. Because they are defined as expectation values in the quantum vacuum these observables all have the interpretation of ensemble averages, as will be discussed in more detail below. On the other hand, it is sometimes required to know the probability that fluctuations of some given magnitude occur in the perturbation \mathcal{R} [37, 14, 38]. This is not a question about ensemble averages, but instead is concerned with the probability measure on the ensemble itself. As a result, such information cannot easily be obtained from inspection or simple manipulation of the n -point functions.

For example, if we know by some a priori means that \mathcal{R} is free, then the argument given above based on the central limit theorem implies that at any position \mathbf{x} , the probability of fluctuations in \mathcal{R} of size ϵ must be

$$\mathbf{P}(\mathcal{R} \text{ has fluctuations of size } \epsilon) \simeq \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\epsilon^2}{2\sigma^2}\right), \quad (3)$$

where the variance in \mathcal{R} is

$$\sigma^2 = \langle \mathcal{R}(t, \mathbf{x})^2 \rangle = \int d \ln k \, \mathcal{P}(k). \quad (4)$$

The quantity $\mathcal{P}(k)$ is the so-called dimensionless power spectrum, which is defined in terms of the two-point function of \mathcal{R} , calculated from the quantum field theory in-vacuum:

$$\langle \text{in} | \mathcal{R}(t, \mathbf{k}_1) \mathcal{R}(t, \mathbf{k}_2) | \text{in} \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) \frac{2\pi^2}{k_1^3} \mathcal{P}(k_1). \quad (5)$$

This is the only relevant observable, because it is a standard property of free fields that all other non-vanishing correlation functions can be expressed in terms of the two-

point function (5), and hence the power spectrum. In practice, in order to give a precise meaning to (3), it would be necessary to specify what it means for \mathcal{R} to develop fluctuations of size ϵ , and whether it is the fluctuations in the microphysical field \mathcal{R} or some smoothed field $\bar{\mathcal{R}}$ which are measured. These details affect the exact expression (4) for the variance of ϵ .

The average in Eq. (5), denoted by $\langle \text{in} | \cdots | \text{in} \rangle$, is the expectation value in the quantum in-vacuum. To relate this abstract expectation value to real-world measurement probabilities, one introduces a notional ensemble of possible universes, of which the present universe and the density fluctuation that we observe is only one possible realization (see, *eg.*, [39]). However, for ergodic processes, we may freely trade ensemble averages for volume averages. If we make the common supposition that the inflationary density perturbation is indeed ergodic, then we expect the volume average of the density fluctuation to behave like the ensemble average: the universe may contain regions where the fluctuation is atypical, but with high probability most regions contain fluctuations of root mean square (rms) amplitude close to σ . Therefore the probability distribution on the ensemble, which is encoded in Eq. (5), translates to a probability distribution on smoothed regions of order the horizon size within our own universe.

In order to apply the above analysis, it was necessary to know in advance that \mathcal{R} was a free field. This knowledge allowed us to use the central limit theorem to connect the correlation functions of \mathcal{R} with the probability distribution (3). The situation in the real universe is not so simple. In particular, the assumption that during inflation \mathcal{R} behaves as a free field, and therefore that the oscillators $\mathcal{R}(\mathbf{k})$ are uncorrelated and independently distributed, is only approximately correct. In fact, \mathcal{R} is subject to self-interactions and interactions with the other constituents of the universe, which mix \mathbf{k} -modes. Consequently, the oscillators $\mathcal{R}(\mathbf{k})$ acquire some phase correlation and are no longer independently distributed. In this situation the central limit theorem gives only approximate information concerning the probability distribution of $\mathcal{R}(\mathbf{x})$, and it is necessary to use a different method to connect the correlation functions of \mathcal{R} with its probability distribution.

In this paper we give a new derivation of the probability distribution of the amplitude of fluctuations in \mathcal{R} which directly connects $\mathbf{P}(\epsilon)$ and the correlation functions $\langle \mathcal{R}(\mathbf{k}_1) \cdots \mathcal{R}(\mathbf{k}_n) \rangle$, without intermediate steps which invoke the central limit theorem or other statistical results. When the inflaton is treated as a free field, our method reproduces the familiar prediction (3) of Gaussian statistics. When the inflaton is *not* treated as a free field, the very significant advantage of our technique is that it is possible to directly calculate the corrections to $\mathbf{P}(\epsilon)$. Specifically, the interactions of \mathcal{R} can be measured by the departure of the correlation functions from the form they would take if \mathcal{R} were free. Therefore, the first corrections to the free-field approximation are contained in the three-point function, which is exactly zero when there are no interactions.

The three-point function for single-field, slow-roll inflation has been calculated by Maldacena [40], whose result can be expressed in the form [41]

$$\langle \mathcal{R}(\mathbf{k}_1) \mathcal{R}(\mathbf{k}_2) \mathcal{R}(\mathbf{k}_3) \rangle = 4\pi^4 (2\pi)^3 \delta\left(\sum_i \mathbf{k}_i\right) \frac{\bar{\mathcal{P}}^2}{\prod_i k_i^3} \mathcal{A}(k_1, k_2, k_3), \quad (6)$$

where \mathcal{A} is one-half§ Maldacena's \mathcal{A} -function [40] and $\bar{\mathcal{P}}^2$ (to be defined later) measures the amplitude of the spectrum when the \mathbf{k}_i crossed the horizon. (For earlier work, see Refs. [43, 44, 45, 46]. The present situation is reviewed in Ref. [47].) This result has since been extended to cover the non-Gaussianity produced during slow-roll inflation with an arbitrary number of fields [40, 41, 48, 49, 50, 51, 52, 53], preheating [54, 55, 56], models where the dominant non-Gaussianity is produced by a light scalar which is a spectator during inflation [57, 58, 39], and alternative models, often involving a small speed of sound for the inflaton perturbation [42, 59, 60, 61, 48]. For single-field, slow-roll inflation, the self-interactions of \mathcal{R} are suppressed by powers of the slow-roll parameters. This means that the correction to Gaussian statistics is not large. In terms of the \mathcal{A} -parametrized three-point function (6), this is most commonly expressed by writing

$$f_{\text{NL}} = -\frac{5}{6} \frac{\mathcal{A}}{\sum_i k_i^3} = \text{O}(\text{slow roll}), \quad (7)$$

where f_{NL} [28, 62] expresses the relative contribution of a non-Gaussian piece in \mathcal{R} , viz, $\mathcal{R} = \mathcal{R}_g - \frac{3}{5} f_{\text{NL}} \star \mathcal{R}_g^2$, and \mathcal{R}_g is a Gaussian random field. (There are differing sign conventions for f_{NL} [39].) In models with more degrees of freedom it is expected to be possible to obtain very much larger non-Gaussianities, perhaps with $f_{\text{NL}} \sim 10$ [63, 64, 65, 66, 57, 50, 53], although no unambiguous concrete example yet exists except where the non-Gaussianity is generated during preheating [54, 56]. If the inflationary perturbation has a speed of sound different from unity then large non-Gaussianities may also appear [42], although in this case it is difficult to simultaneously achieve scale invariance. The current observational constraint can approximately be expressed as $|f_{\text{NL}}| \lesssim 100$ [1]. In the absence of a detection, the forthcoming Planck Explorer mission may tighten this constraint to $|f_{\text{NL}}| \lesssim 3$ [28, 34].

Non-Gaussian probability distributions have been studied previously by several authors. The closest to the method developed in this paper include that of Matarrese, Verde & Riotto [67], who worked with a path integral expression for the density fluctuation smoothed on a scale R (which they denoted ' δ_R '); and that of Bernardeau & Uzan [68, 69]. The latter analysis has some features in common with our own, being based on the cumulant generating function, and moreover since the expression for the probability density in Refs. [68, 69] is expressed as a Laplace transform. Our final expression, Eq. (65), can be interpreted as a Fourier integral, viz (15), which (loosely

§ In Maldacena's normalization, the numerical prefactor in Eq. (6) is not consistent with the square of the two-point function, Eq. (5). We choose \mathcal{A} so that the prefactor becomes $4\pi^2(2\pi)^3$. This normalization of Eq. (6) was also employed in Refs. [42, 41], although the distinction from Maldacena's \mathcal{A} was not pointed out explicitly. Throughout we work in units where the reduced Planck mass, defined by $M_{\text{P}}^2 = (8\pi G)^{-1}$, is set to unity. If necessary, a finite Planck mass can be restored in formulas such as (6) by dimensional analysis.

speaking) can be related to a Laplace integral via a Wick transformation. Despite these similarities, the correspondence between the two is complicated because Refs. [68, 69] work in a multiple-field picture and calculate a probability density only for the isocurvature field ‘ δs ’, which acquires its non-Gaussianity via a mixing of isocurvature and adiabatic modes long after horizon exit. This contrasts with the situation in the present paper, where we restrict ourselves to a single-field scenario and compute the probability density for the adiabatic mode \mathcal{R} , which would be orthogonal to δs in field space and whose non-Gaussianity is generated exactly at horizon exit.

In the older literature it is more common to deal with the density fluctuation $\delta\rho/\rho$ measured on comoving slices, as an alternative to the curvature perturbation \mathcal{R} . For slowly varying fields, on large scales, \mathcal{R} and $\delta\rho$ can be related via the rule (Eq. (25) of Ref. [70]),

$$\left(\frac{aH}{k}\right)^2 \frac{\delta\rho}{\rho} = -\left(\frac{3}{2} + \frac{1}{1+\omega}\right)^{-1} \mathcal{R}, \quad (8)$$

which is valid outside the horizon, and to first order in cosmological perturbation theory, for a fluid with equation of state $p = \omega\rho$. (One may use the δN formalism to go beyond leading order, but to obtain results valid on sub-horizon scales one must use the full Einstein equations directly; see, *eg.*, [71, 72].) For fluctuations on the Hubble scale, where $k \simeq aH$, this means $|\mathcal{R}| \simeq \delta\rho/\rho$, so \mathcal{R} provides a useful measure of the density fluctuation on such scales. In virtue of this relationship with the density fluctuation, the probability distribution $\mathbf{P}(\epsilon)$ is an important theoretical tool, especially in studies of structure formation. For example, it is the principal object in the Press–Schechter formalism [37]. As a result, there are important reasons why knowledge of the detailed form of the probability distribution of ϵ is important, and not merely the approximate answer provided by the central limit theorem.

Firstly, large collapsed objects, such as primordial black holes (PBHs) naturally form on the high- ϵ tail of the distribution [73, 74]. Such large fluctuations are extremely rare. This means that a small change in the probability density for $|\epsilon| \gg 0$ can make a large difference in the mass fraction of the universe which collapses into PBHs [75, 76]. Thus one may hope to probe it using well-known and extremely stringent constraints on PBH formation in the early universe [77, 78]. The corrections calculated in this paper are therefore not merely of theoretical interest, but relate directly to observations, and have the potential to sharply discriminate between models of inflation.

Secondly, as described above, although the non-Gaussianities produced by single-field, slow-roll inflation are small, this is not mandatory. In models where non-Gaussianities are large, it will be very important to account for the effect of non-Gaussian fluctuations on structure formation [62, 67, 79, 80]. The formalism presented in this paper provides a systematic way to obtain such predictions, extending the analysis given in Ref. [67].

The outline of this paper is as follows. In Section 2 we obtain the probability measure on the ensemble of possible fluctuations. This step depends on the correlation functions of \mathcal{R} . In Section 3, we discuss the decomposition of \mathcal{R} into harmonics. This

is a technical step, which is necessary in order to write down a path integral for $\mathbf{P}(\epsilon)$. First, \mathcal{R} is decomposed in Section 3.1. The path integral measure is written down in Section 3.2, and in Section 3.3 we give a precise specification of ϵ , which measures the size of fluctuations. We distinguish two interesting cases, a “total fluctuation” ϵ , which corresponds to \mathcal{R} (or approximately $\delta\rho/\rho$) smoothed over regions the size of the Hubble volume, and the “spectrum” $\varrho(k)$, which describes the contributions to ϵ from regions of the primordial power spectrum around the scale described by wavenumber k . In Section 4 we evaluate $\mathbf{P}(\epsilon)$. We give the calculation for the Gaussian case first, in Section 4.1, in order to clearly explain our method with a minimum of technical distractions. This is followed in Section 4.2 by the same calculation but including non-Gaussian corrections which follow from a non-zero three-point function. In Section 5 we calculate $\mathbf{P}[\varrho(k)]$. Finally, we state our conclusions in Section 6.

2. The probability measure on the ensemble of \mathcal{R}

Our method is to compute the probability measure $\mathbf{P}_t[\mathbf{R}]$ on the ensemble of realizations of the curvature perturbation $\mathbf{R}(\mathbf{x})$, which we define to be the value of $\mathcal{R}(t, \mathbf{x})$ at some fixed time t . This probability measure is a natural object in the Schrödinger approach to quantum field theory, where the elementary quantity is the wavefunctional $\Psi_t[\mathbf{R}]$, which is related to $\mathbf{P}_t[\mathbf{R}]$ by the usual rule of quantum mechanics, that $\mathbf{P}_t[\mathbf{R}] \propto |\Psi_t[\mathbf{R}]|^2$. Once the measure $\mathbf{P}_t[\mathbf{R}]$ is known, we can directly calculate (for example) $\mathbf{P}_t(\epsilon)$ by integrating over all \mathbf{R} that produce fluctuations of amplitude ϵ . Although the concept of a probability measure on \mathbf{R} may seem like a rather formal object, the Schrödinger representation^{||} of quantum field theory is entirely equivalent to the more familiar formulation in terms of a Fock space. Indeed, a similar procedure has been discussed by Ivanov [76], who calculated the probability measure on a stochastic metric variable $a_{\text{ls}}(\mathbf{x})$ which can be related to our $\mathcal{R}(\mathbf{x})$. Although the approaches are conceptually similar, our method is substantially different in detail. In particular, the present calculation is exact in the sense that we make no reference to the stochastic approach to inflation, and therefore are not obliged to introduce a coarse-graining approximation. Moreover, Ivanov’s analysis appeared before the complete non-Gaussianity arising from \mathcal{R} -field interactions around the time of horizon crossing had been calculated [40], and therefore did not include this effect.

2.1. The generating functional of correlation functions

The expectation values $\langle \mathcal{R}(\mathbf{x}_1) \cdots \mathcal{R}(\mathbf{x}_2) \rangle$ in the vacuum $|\text{in}\rangle$ at some fixed time t can be expressed in terms of a Schwinger–Keldysh path integral,[¶]

$$\langle \mathcal{R}(t, \mathbf{x}_1) \cdots \mathcal{R}(t, \mathbf{x}_n) \rangle =$$

^{||} The Schrödinger representation is briefly discussed in, for example, Refs [81, 82]. A brief introduction to infinite-dimensional probability measures is given in Ref. [83].

[¶] Henceforth, we use the notation $\langle \cdots \rangle$ to mean expectation values in the in-vacuum, and no longer write $|\text{in}\rangle$ explicitly where this is unambiguous.

$$\int [d\mathcal{R}_- d\mathcal{R}_+]_{|\text{in}}^{\mathcal{R}_+(t,\mathbf{x})=\mathcal{R}_-(t,\mathbf{x})} \mathcal{R}(t, \mathbf{x}_1) \cdots \mathcal{R}(t, \mathbf{x}_n) \exp(iS[\mathcal{R}_+] - iS[\mathcal{R}_-]) . \quad (9)$$

(For details of the Schwinger–Keldysh or “closed time path” formalism, see Refs. [17, 18, 23, 21, 20, 22].) In cosmology we are generally interested in \mathcal{R} evaluated at different spatial positions on the same t -slice, so we have set all the t equal in (9). The path integral is taken over all fields \mathcal{R} which begin in a configuration corresponding to the vacuum $|\text{in}\rangle$ at past infinity, and $S[\mathcal{R}]$ is the action for the fluctuation \mathcal{R} , which is given to third order in \mathcal{R} in Refs. [40, 42, 41].

An expression equivalent to Eq. (9) can be given in terms of the “equal time” generating functional $Z_t[\eta]$ at time t , defined by

$$Z_t[\eta] = \int [d\mathcal{R}] \int [d\mathcal{R}_- d\mathcal{R}_+]_{|\text{in}}^{\mathcal{R}_\pm(t,\mathbf{x})=\mathcal{R}(\mathbf{x})} \exp\left(iS[\mathcal{R}_+] - iS[\mathcal{R}_-] + i \int_{\Sigma_t} d^3x \mathcal{R}(\mathbf{x}) \eta(\mathbf{x})\right), \quad (10)$$

where η is some arbitrary source field and Σ_t is a spatial slice at coordinate time t . This generating functional is *not* the usual one, which would generate correlation function at any given set of times, and not the common time t which appears in (9)–(10). The equal-time correlation functions $\langle \mathcal{R}(t, \mathbf{x}_1) \cdots \mathcal{R}(t, \mathbf{x}_n) \rangle$ are recovered from $Z_t[\eta]$ by functional differentiation,

$$\langle \mathcal{R}(t, \mathbf{x}_1) \cdots \mathcal{R}(t, \mathbf{x}_n) \rangle = \frac{1}{i^n} \frac{\delta}{\delta \eta(\mathbf{x}_1)} \cdots \frac{\delta}{\delta \eta(\mathbf{x}_n)} \ln Z_t[\eta] \Big|_{\eta=0}. \quad (11)$$

Up to normalization, this is merely the rule for functional Taylor coefficients, so it is straightforward to invert Eq. (11) for $Z_t[\eta]$. We obtain⁺

$$Z_t[\eta] = \exp \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \cdots \int d^3x_1 \cdots d^3x_n \eta(\mathbf{x}_1) \cdots \eta(\mathbf{x}_n) \langle \mathcal{R}(t, \mathbf{x}_1) \cdots \mathcal{R}(t, \mathbf{x}_n) \rangle, \quad (12)$$

Eq. (10) for the generating functional can be rewritten in a suggestive way. We define the wavefunctional at time t , $\Psi_t[\mathcal{R}]$, as

$$\Psi_t[\mathcal{R}] = \int [d\mathcal{R}]_{|\text{in}}^{\mathcal{R}(t,\mathbf{x})=\mathcal{R}(\mathbf{x})} \exp(iS[\mathcal{R}]). \quad (13)$$

This definition is simply the functional generalization of the familiar quantum-mechanical wavefunction. It expresses the amplitude for the field $\mathcal{R}(t, \mathbf{x})$ to have the spatial configuration $\mathcal{R}(\mathbf{x})$ at time t , given the boundary condition that \mathcal{R} started in the vacuum state in the far past. In terms of $\Psi_t[\mathcal{R}]$, the generating functional can be rewritten

$$Z_t[\eta] = \int [d\mathcal{R}] \Psi_t[\mathcal{R}]^\dagger \Psi_t[\mathcal{R}] \exp\left(i \int d^3x \mathcal{R}(\mathbf{x}) \eta(\mathbf{x})\right) = |\widetilde{\Psi_t[\mathcal{R}]}|^2 \propto \widetilde{\mathbf{P}[\mathcal{R}]}, \quad (14)$$

⁺ This construction is somewhat similar to the dS/CFT calculations outlined in Refs. [40, 84, 85, 86]. In these calculations one constructs $|\Psi|^2$ from an expression of the same form as (12), but expressed in terms of an operator \mathcal{O} which is the holographic dual of the bulk field \mathcal{R} . The dS/CFT prescription relates the correlators of \mathcal{O} reciprocally to those of \mathcal{R} . In this paper we do not make any use of holographic arguments, but the same reciprocal relationship emerges through the identification of $Z_t[\eta]$ with the Fourier transform of $|\Psi|^2$. The normal rules of Fourier transforms show that the transform of a Gaussian is another Gaussian with a reciprocal coefficient, $e^{-ax^2/2} \xrightarrow{\text{Fourier}} e^{-k^2/2a}$.

where a tilde denotes a (functional) Fourier transform, and \dagger denotes Hermitian conjugation. Eq. (14) implies that $Z_t[\eta]$ is the complementary function [83] for the probability distribution $\mathbf{P}_t[\mathbf{R}]$, which can formally be obtained by inversion of $Z_t[\eta]$. Hence, up to an overall normalization,

$$\mathbf{P}_t[\mathbf{R}] \propto \int [d\eta] \exp \left(-i \int d^3x \mathbf{R}(\mathbf{x}) \eta(\mathbf{x}) \right) Z_t[\eta]. \quad (15)$$

The normalization is not determined by this procedure, but it is irrelevant. We will fix the \mathbf{R} -independent prefactor, which correctly normalizes the probability distribution, by requiring $\int d\epsilon \mathbf{P}(\epsilon) = 1$ at the end of the calculation. For this reason, we systematically drop all field-independent prefactors in the calculation which follows.

2.2. The probability density on the ensemble

So far, all our considerations have been exact, and apply for any quantum field $\mathcal{R}(t, \mathbf{x})$. For any such field, Eq. (15) gives the probability density for a spatial configuration \mathbf{R} at time t , and implies that to obtain $\mathbf{P}_t[\mathbf{R}]$ we should need to know *all* such functions for all n -point correlations, and at all spatial positions \mathbf{x} . In practice, some simplifications occur when \mathcal{R} is identified as the inflationary curvature perturbation.

The most important simplification is the possibility of a perturbative evaluation. The dominant mode of the CMB fluctuation is constrained to be Gaussian to high accuracy, so the corrections to the leading order Gaussian result cannot be large. Moreover, since we assume \mathcal{R} is the curvature perturbation which is communicated to CMB fluctuations, the amplitude of its spectrum is constrained by observation. Specifically, in the region of wavenumbers probed by the COBE DMR instrument [87] (and other later CMB experiments such as BOOMERANG [4] and WMAP [2]), the spectrum has amplitude $\mathcal{P}^{1/2} \sim 10^{-5}$, whereas the requirement that inflation not overproduce PBHs generally requires $\mathcal{P}^{1/2} \lesssim 10^{-3}$ over the relevant wavenumbers [77]. Each higher-order correlation function is suppressed by an increasing number of copies of the spectrum, $\mathcal{P}(k)$. Provided the amplitude of \mathcal{P} is small, it is reasonable to believe that we are justified in truncating the exponential in (15) at a given level in n and working with a perturbation series in \mathcal{P} . The relevant criterion is the smallness of the spectrum, rather than the validity of slow-roll.

However, this simple approach is too naïve, because the integrals over η eventually make any given term in the series large, and invalidate simple perturbative arguments based on power counting in \mathcal{P} . The perturbation series can only be justified *a posteriori*, a point to which we will return in Section 4.2.

We work to first-order in the three-point correlation,

$$\mathbf{P}_t[\mathbf{R}] \propto \int [d\eta] \Upsilon_t[\eta] \omega_t[\eta; \mathbf{R}], \quad (16)$$

where $\Upsilon[\eta]$ and $\omega[\eta; \mathbf{R}]$ are defined by

$$\Upsilon_t[\eta] = \left(1 - \frac{i}{6} \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} \eta(\mathbf{k}_1) \eta(\mathbf{k}_2) \eta(\mathbf{k}_3) \langle \mathcal{R}(t, \mathbf{k}_1) \mathcal{R}(t, \mathbf{k}_2) \mathcal{R}(t, \mathbf{k}_3) \rangle \right), \quad (17)$$

and

$$\omega_t[\eta; \mathbf{R}] = \exp \left(- \int \frac{d^3 k_1 d^3 k_2}{(2\pi)^6} \frac{\eta(\mathbf{k}_1)\eta(\mathbf{k}_2)}{2} \langle \mathcal{R}(t, \mathbf{k}_1) \mathcal{R}(t, \mathbf{k}_2) \rangle - i \int \frac{d^3 k}{(2\pi)^3} \eta(\mathbf{k}) \mathbf{R}(\mathbf{k}) \right). \quad (18)$$

In this expression, ω will give rise to the Gaussian part of the probability distribution, and Υ is of the form 1 plus a correction. This correction is small when the perturbative analysis is valid. Higher-order perturbative corrections in \mathcal{P} can be accommodated if desired by retaining higher-order terms in the power-series expansion of the exponential in (15). Therefore our method is not restricted to corrections arising from non-Gaussianities described by three-point correlations only, but can account for non-Gaussianities which enter at any order in the correlations of \mathcal{R} , limited only by the computational complexity. However, in this paper, we work only with the three-point non-Gaussianity.

We now complete the square in $\omega_t[\eta; \mathbf{R}]$ and make the finite field redefinition

$$\eta(\mathbf{k}) \mapsto \hat{\eta}(\mathbf{k}) = \eta(\mathbf{k}) + (2\pi)^3 i \frac{\mathbf{R}(\mathbf{k})}{\langle \mathcal{R}(t, \mathbf{k}) \mathcal{R}(t, -\mathbf{k}) \rangle'}, \quad (19)$$

where the prime ' attached to $\langle \mathcal{R}(t, \mathbf{k}) \mathcal{R}(t, -\mathbf{k}) \rangle'$ indicates that the momentum-conservation δ -function is omitted. The measure $[d\eta]$ is formally invariant under this shift, giving $\int [d\eta] = \int [d\hat{\eta}]$, whereas $\omega_t[\eta; \mathbf{R}]$ can be split into an \mathbf{R} -dependent piece, which we call $\Gamma_t[\mathbf{R}]$, and a piece that depends only on $\hat{\eta}$ but not \mathbf{R} ,

$$\omega_t[\eta; \mathbf{R}] \mapsto \Gamma_t[\mathbf{R}] \exp \left(- \frac{1}{2} \int \frac{d^3 k_1 d^3 k_2}{(2\pi)^6} \hat{\eta}(\mathbf{k}_1) \hat{\eta}(\mathbf{k}_2) \langle \mathcal{R}(t, \mathbf{k}_1) \mathcal{R}(t, \mathbf{k}_2) \rangle \right), \quad (20)$$

where $\Gamma_t[\mathbf{R}]$ is a Gaussian in \mathbf{R} ,

$$\Gamma_t[\mathbf{R}] = \exp \left(- \frac{1}{2} \int d^3 k_1 d^3 k_2 \langle \mathcal{R}(t, \mathbf{k}_1) \mathcal{R}(t, \mathbf{k}_2) \rangle \frac{\mathbf{R}(\mathbf{k}_1) \mathbf{R}(\mathbf{k}_2)}{\prod_i \langle \mathcal{R}(t, \mathbf{k}_i) \mathcal{R}(t, -\mathbf{k}_i) \rangle'} \right). \quad (21)$$

Eq. (16) for the probability density becomes

$$\mathbf{P}_t[\mathbf{R}] \propto \Gamma_t[\mathbf{R}] \int [d\hat{\eta}] \Upsilon_t \exp \left(- \frac{1}{2} \int \frac{d^3 k_1 d^3 k_2}{(2\pi)^6} \hat{\eta}(\mathbf{k}_1) \hat{\eta}(\mathbf{k}_2) \langle \mathcal{R}(t, \mathbf{k}_1) \mathcal{R}(t, \mathbf{k}_2) \rangle \right), \quad (22)$$

One can easily verify that this is the correct expression, since if we ignore the three-point contribution (thus setting $\Upsilon_t = 1$), one recovers (after applying a correct normalization)

$$\int [d\mathbf{R}] \mathbf{R}(\mathbf{k}_1) \mathbf{R}(\mathbf{k}_2) \Gamma_t[\mathbf{R}] = \langle \mathcal{R}(t, \mathbf{k}_1) \mathcal{R}(t, \mathbf{k}_2) \rangle. \quad (23)$$

The remaining issue is to carry out the $\hat{\eta}$ integrations in Υ_t . The only terms which contribute are those containing an even power of $\hat{\eta}$, since any odd function integrated against $e^{-\hat{\eta}^2}$ vanishes identically. In the expansion of $\prod_i \eta(\mathbf{k}_i)$ in terms of $\hat{\eta}$, there are two such terms: one which is quadratic in $\hat{\eta}$, and one which is independent of $\hat{\eta}$. These are accompanied by linear and cubic terms which do not contribute to $\mathbf{P}_t[\mathbf{R}]$. For any symmetric kernel \mathbf{K} and vectors $\mathbf{p}, \mathbf{q} \in \mathbf{R}^m$, one has the general results

$$\int [df] \exp \left(- \frac{1}{2} \int d^m x d^m y f(\mathbf{x}) f(\mathbf{y}) \mathbf{K}(\mathbf{x}, \mathbf{y}) \right) = (\det \mathbf{K})^{-1/2}, \quad (24)$$

$$\int [df] f(\mathbf{p}) f(\mathbf{q}) \exp \left(- \frac{1}{2} \int d^m x d^m y f(\mathbf{x}) f(\mathbf{y}) \mathbf{K}(\mathbf{x}, \mathbf{y}) \right) = \mathbf{K}^{-1}(\mathbf{p}, \mathbf{q}) (\det \mathbf{K})^{-1/2}. \quad (25)$$

These rules allow us to evaluate the $\hat{\eta}$ integrals in Eq. (22), giving

$$\mathbf{P}_t[\mathbf{R}] \propto \Gamma_t[\mathbf{R}] \left(1 + \Upsilon_t^{(0)}[\mathbf{R}] + \Upsilon_t^{(2)}[\mathbf{R}] \right), \quad (26)$$

where $\Upsilon^{(0)}$ and $\Upsilon^{(2)}$ are defined by

$$\Upsilon_t^{(0)}[\mathbf{R}] = -\frac{1}{6} \int d^3k_1 d^3k_2 d^3k_3 \langle \mathcal{R}(t, \mathbf{k}_1) \mathcal{R}(t, \mathbf{k}_2) \mathcal{R}(t, \mathbf{k}_3) \rangle \frac{R(\mathbf{k}_1) R(\mathbf{k}_2) R(\mathbf{k}_3)}{\prod_i \langle \mathcal{R}(t, \mathbf{k}_i) \mathcal{R}(t, -\mathbf{k}_i) \rangle'}, \quad (27)$$

$$\begin{aligned} \Upsilon_t^{(2)}[\mathbf{R}] = & \frac{1}{6} \int d^3k_1 d^3k_2 d^3k_3 \langle \mathcal{R}(t, \mathbf{k}_1) \mathcal{R}(t, \mathbf{k}_2) \mathcal{R}(t, \mathbf{k}_3) \rangle \frac{R(\mathbf{k}_1) \delta(\mathbf{k}_2 + \mathbf{k}_3)}{\prod_{i \neq 3} \langle \mathcal{R}(t, \mathbf{k}_i) \mathcal{R}(t, -\mathbf{k}_i) \rangle'} \\ & + \text{combinations}, \end{aligned} \quad (28)$$

where in $\Upsilon_t^{(2)}$ we include the possible combinations of the labels $\{1, 2, 3\}$ which give rise to distinct integrands.

In fact, $\Upsilon_t^{(2)}$ is negligible. This happens because the three-point function contains a momentum-conservation δ -function, $\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$, which requires that the vectors \mathbf{k}_i sum to a triangle in momentum space. [For this reason, it is often known as the “triangle condition”, and we will usually abbreviate it schematically as $\delta(\triangle)$.] In combination with the δ -function $\delta(\mathbf{k}_2 + \mathbf{k}_3)$, the effect is to constrain two of the momenta (in this example \mathbf{k}_2 and \mathbf{k}_3) to be equal and opposite, and the other momentum (in this example, \mathbf{k}_1) to be zero. This is the extreme *squeezed* limit [40, 88, 89], in which the bispectrum reduces to the power spectrum evaluated on a perturbed background, which is sourced by the zero-momentum mode. Written explicitly, $\Upsilon_t^{(2)}$ behaves like

$$\Upsilon_t^{(2)}[\mathbf{R}] \simeq \frac{1}{6} \int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \alpha R(\mathbf{k}_1) \delta(\mathbf{k}_1) + \text{combinations}, \quad (29)$$

where we have written $\lim_{k_1 \rightarrow 0} \mathcal{A} = \alpha k_2^3$, and α is a known finite quantity. In particular, Eq. (29) vanishes, provided $R(\mathbf{k})$ approaches zero as $k \rightarrow 0$. This condition is typically satisfied, since by construction $R(\mathbf{k})$ should not contain a zero mode. Indeed, any zero mode, if present, would constitute part of the zero-momentum background, and not a part of the perturbation R . Accordingly, Eqs. (26)–(27) with $\Upsilon_t^{(2)} = 0$ give $\mathbf{P}_t[\mathbf{R}]$ explicitly in terms of the two- and three-point correlation functions.

2.3. The smoothed curvature perturbation

The probability density $\mathbf{P}_t[\mathbf{R}] \propto (1 + \Upsilon_t^{(0)}[\mathbf{R}]) \Gamma_t[\mathbf{R}]$ that we have derived relates to the microphysical field $\mathcal{R}(t, \mathbf{x})$ which appeared in the quantum field theory Lagrangian. A given \mathbf{k} -mode of this field begins in the vacuum state at $t \rightarrow -\infty$. At early times, the mode is far inside the horizon ($k \gg aH$). In this (“subhorizon”) régime, the \mathbf{k} -mode cannot explore the curvature of spacetime and is immune to the fact that it is living in a de Sitter universe. It behaves like a Minkowski space oscillator. At late times, the mode is far outside the horizon ($k \ll aH$). In this (“superhorizon”) régime, the \mathbf{k} -mode asymptotes to a constant amplitude, provided that only one field is dynamically relevant during inflation [90, 13].*

* Where multiple fields are present, there will typically be an isocurvature perturbation between them: hypersurfaces of constant pressure and density will not coincide. Under these circumstances [13], \mathcal{R}

under reasonable conditions the integrals which define the expectation values of \mathcal{R} are typically [23, 24] dominated by the intermediate (“horizon crossing”) régime, where $\mathcal{R}(\mathbf{k})$ is exiting the horizon ($k \sim aH$). As a result, the correlation functions generally depend only on the Hubble and slow-roll parameters around the time of horizon exit.

The simple superhorizon behaviour of \mathcal{R} means that we can treat the power spectrum as constant outside the horizon. As has been described, its value depends only on the Hubble parameter and the slow-roll parameters around the time that the mode corresponding to k exited the horizon. For this reason, the time t at which we evaluate the wavefunctional $\Psi_t[\mathbf{R}]$, the generating functional $Z_t[\eta]$ and the probability distribution $\mathbf{P}_t[\mathbf{R}]$ is irrelevant, provided it is taken to be late enough that the curvature perturbation on interesting cosmological scales has already been generated and settled down to its final value. Indeed, we have implicitly been assuming that t is comoving time, so that observers on slices of constant t see no net momentum flux. Because \mathcal{R} is a gauge invariant, and is constant outside the horizon, our formalism is independent of how we choose to label the spatial slices. The evolution of \mathcal{R} outside the horizon is the principal obstacle which would be involved in extending our analysis to the multiple-field scenario.

When calculating the statistics of density fluctuations on some given lengthscale $2\pi/k_H$, one should smooth the perturbation field over wavenumbers larger than k_H . To take account of this, we introduce a smoothed field $\bar{\mathbf{R}}$ which is related to \mathbf{R} via the rule $\bar{\mathbf{R}}(\mathbf{k}) = \mathcal{W}(k, k_H)\mathbf{R}(\mathbf{k})$, where \mathcal{W} is some window function. The probabilities we wish to calculate and compare to the real universe relate to $\bar{\mathbf{R}}$ rather than \mathbf{R} . The exact choice of filter \mathcal{W} is mostly arbitrary. For the purpose of analytical calculations, it is simplest to pick a sharp cutoff in \mathbf{k} -space, which removes all modes with $k > k_H$. However, this choice has the disadvantage that it is non-local and oscillatory in real space, which makes physical interpretations difficult. The most common alternative choices, which do not suffer from such drawbacks, are: (i) a Gaussian, or: (ii) the so-called “top hat,” which is a sharp cutoff in real space. We allow for a completely general choice of C^0 function \mathcal{W} , subject to the restriction that $\mathcal{W} \neq 0$ except at $k = \infty$ and possibly at an isolated set of points elsewhere. This restriction is made so that there is a one-to-one relationship between $\bar{\mathbf{R}}$ and \mathbf{R} . If this were not the case, it would be necessary to coarse-grain over classes of microphysical fields \mathbf{R} which would give rise to the same smoothed field $\bar{\mathbf{R}}$.

In addition to this smoothing procedure, the path integral must be regulated before carrying out the calculation in the next section. This is achieved by artificially compactifying momentum space, so that the range of available wavenumbers is restricted to $k < \Lambda$, where Λ is an auxiliary hard cutoff.[‡] At the end of the calculation one will evolve. We do not consider the evolving case in this paper, but rather restrict our attention to the single-field case where the superhorizon behaviour of \mathcal{R} is simple.

[‡] Note that this procedure does not have anything to do with the regularization of ultraviolet divergences. Such divergences are connected to the appearance of loop graphs, which we ignore, and in any case are subdominant [23, 24].

takes $\Lambda \rightarrow \infty$. Some care is necessary in carrying out this compactification. We set $\bar{R} = 0$ for $k > \Lambda$. In order to maintain continuity at $k = \Lambda$, we introduce a 1-parameter family of functions \mathcal{W}_Λ . These functions are supposed to satisfy the matching condition $\lim_{\Lambda \rightarrow \infty} \mathcal{W}_\Lambda(k) = \mathcal{W}(k)$, and are subject to the restriction $\mathcal{W}_\Lambda(\Lambda) = 0$. (These conditions could perhaps be relaxed.) The relationship between R and \bar{R} becomes

$$\bar{R}(\mathbf{k}) = \theta(\Lambda - k) \mathcal{W}_\Lambda(k; k_H) R(\mathbf{k}). \quad (30)$$

where $\theta(x) = \int_{-\infty}^x \delta(z) dz$ is the Heaviside function. To minimise unnecessary clutter in equations, we frequently suppress the Λ and k_H dependence in \mathcal{W} , writing only $\mathcal{W}(k)$ with the smoothing scale k_H and hard cutoff Λ left implicit. Both of the standard window functions approach zero as $k \rightarrow \infty$, and are compatible with (30) in the $\Lambda \rightarrow \infty$ limit. In this limit, the final result is independent of the exact choice of family $\mathcal{W}_\Lambda(k, k_H)$.

The probability in which we are interested is that of observing a given filtered field \bar{R} . One can express this via the rule (see also [67, 91])

$$\mathbf{P}_t[\bar{R}] = \int [dR] \mathbf{P}_t[R] \delta[\bar{R} = \theta(\Lambda - k) \mathcal{W} R]. \quad (31)$$

3. Harmonic decomposition of the curvature perturbation

In the previous section, we obtained the probability density for a given smoothed spatial configuration of the curvature perturbation. Given this probability density, the probability $\mathbf{P}(\zeta)$ that the configuration exhibits some characteristic ζ , such as fluctuations of size ϵ or a ‘fluctuation spectrum’ (to be defined later) of the form $\varrho(k)$, is formally obtained integrating over all configurations of \bar{R} which exhibit the criteria which define ζ (cf. Ref. [67]). We give a precise specification of these criteria in Section 3.3 below. Before doing so, however, we exploit the compactification of momentum space introduced in (30) to define a complete set of partial waves. The smoothed field \bar{R} can be written as a superposition of these partial waves with arbitrary coefficients. Moreover, the path integral measure can formally be written as a product of conventional integrals over these coefficients [92].

In this section we assemble the necessary formulae for the partial-wave decomposition. In particular, we shall require Eq. (32) for the decomposition of \bar{R} , Eq. (38) for the path integral measure, and Eq. (41), which gives a precise specification of the characteristics ϵ and $\varrho(k)$.

3.1. Harmonic expansion of \bar{R}

We expand $\bar{R}(\mathbf{k})$ in harmonics on the unit sphere and along the radial $k = |\mathbf{k}|$ direction,

$$\bar{R}(\mathbf{k}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{n=1}^{\infty} \bar{R}_{\ell n}^m Y_{\ell m}(\theta, \phi) \psi_n(k). \quad (32)$$

The $Y_{\ell m}(\theta, \phi)$ are the standard spherical harmonics on the unit 2-sphere (see, *eg.*, Ref. [93]), whereas the $\psi_n(k)$ are any complete, orthogonal set of functions on the interval $[0, \Lambda]$. These harmonics should satisfy the following conditions:^{††}

- (i) $\psi_n(k) \rightarrow 0$ smoothly as $k \rightarrow 0$, so that power is cut off on very large scales, and the universe remains asymptotically FRW with the zero-mode $a(t)$ which was used when computing the expectation values $\langle \mathcal{R} \cdots \mathcal{R} \rangle$ left intact;
- (ii) $\psi_n(k) \rightarrow 0$ smoothly as $k \rightarrow \Lambda$, so that the resulting $\bar{\mathbf{R}}$ is compatible with Eq. (30);
- (iii) $\psi_n(k)$ should have dimension $[M^{-3}]$, in order that Eq. (32) is dimensionally correct; and
- (iv) the $\psi_n(k)$ should be orthogonal in the measure $\int_0^\Lambda dk k^5 \mathcal{P}^{-1}(k) \mathcal{W}^{-2}(k)$.

In addition, there is a constraint on the coefficients $\bar{\mathbf{R}}_{\ell|n}^m$, because $\bar{\mathbf{R}}(\mathbf{k})$ should be real in configuration space and therefore must obey the Fourier reality condition $\bar{\mathbf{R}}(\mathbf{k})^* = \bar{\mathbf{R}}(-\mathbf{k})$, where a star ‘ $*$ ’ denotes complex conjugation. The $\bar{\mathbf{R}}_{\ell|n}^m$ are generically complex, so it is useful to separate the real and imaginary parts by writing $\bar{\mathbf{R}}_{\ell|n}^m = a_{\ell|n}^m + ib_{\ell|n}^m$. The condition that $\bar{\mathbf{R}}$ is real in configuration space implies

$$a_{\ell|n}^{-m} = (-1)^{\ell+m} a_{\ell|n}^m \quad (33)$$

$$b_{\ell|n}^{-m} = (-1)^{\ell+m+1} b_{\ell|n}^m. \quad (34)$$

These conditions halve the number of independent coefficients, since the a and b coefficients with strictly negative m are related to those with strictly positive m , whereas for the $m = 0$ modes, the b coefficients vanish if ℓ is even, and the a coefficients vanish if ℓ is odd.

Condition (i) is made so that the power on large scales is smoothly cut off. In the absence of this constraint, $\bar{\mathbf{R}}$ could develop unbounded fluctuations on extremely large scales which would renormalize $a(t)$. Therefore, Condition (i) can be interpreted as a consistency requirement, since the inflationary two- and three-point functions are calculated using perturbation theory on an FRW universe with some given $a(t)$, which must be recovered asymptotically as $|\mathbf{x}| \rightarrow \infty$. It will later be necessary to sharpen this condition to include constraints on the behaviour of $\mathcal{P}(k)$ near $k = 0$, beyond the weak requirement that $\sigma^2 = \int \mathcal{P}(k) d \ln k$ is finite. Condition (iv) is a technical condition made for future convenience. Any other choice of normalization would work just as well, but this choice is natural given the k -dependence in the Gaussian kernel $\Gamma[\bar{\mathbf{R}}]$. Indeed, with this condition, the Gaussian prefactor in $\mathbf{P}(\epsilon)$ will reduce to the exponential of a sum of

^{††} When expanding functions on \mathbf{R}^3 in terms of polar coordinates, a more familiar expansion is in terms of the spherical waves $Z_{\ell m|k} \propto j_\ell(kr) Y_{\ell m}(\theta, \phi)$, where j_ℓ is a spherical Bessel function. These waves are eigenfunctions of the Laplacian in polar coordinates, viz, $\nabla^2 Z_{\ell m|k} = -k^2 Z_{\ell m|k}$. An arbitrary function on \mathbf{R}^3 can be written in terms of the spherical waves, which is equivalent to a Fourier expansion. We do not choose the spherical waves as an appropriate complete, orthogonal set of basis functions because we do not wish to expand *arbitrary* functions, but rather functions obeying particular boundary conditions, specifically, at $k = 0$. The spherical waves for low ℓ behave improperly at small k for this purpose. Moreover, it is not possible to easily impose the boundary condition $\bar{\mathbf{R}}(k) \rightarrow 0$ as $k \rightarrow \Lambda$.

squares of the $a_{\ell n}^m$ and $b_{\ell n}^m$. We stress that in virtue of Condition (iii), the inner product of two $\psi_n(k)$ in the measure $\int_0^\Lambda dk k^5 \mathcal{P}^{-1}(k) \mathcal{W}^{-2}(k)$ is dimensionless. Condition (ii) has less fundamental significance. It follows from the condition $\mathcal{W}_\Lambda(\Lambda) = 0$ and the artificial compactification of momentum space. However, as in the usual Sturm–Liouville theory [94], the precise choice of boundary condition is immaterial when the regulator is removed and $\Lambda \rightarrow \infty$. Condition (ii) does not affect the final answer.

To demonstrate the existence of a suitable set of $\psi_n(k)$, we can adopt the definition

$$\psi_n(k) = \frac{\sqrt{2}}{J_{\nu+1}(\alpha_\nu^n)} \frac{\mathcal{P}(k) \mathcal{W}(k)}{\Lambda k^2} J_\nu \left(\alpha_\nu^n \frac{k}{\Lambda} \right), \quad (35)$$

where $J_\nu(z)$ is the Bessel function of order ν , which is regular at the origin, and α_ν^n is its n th zero. The order ν is arbitrary, except that in order to obey condition (i) above, we must have $k^{\nu-2} \mathcal{P}(k) \rightarrow 0$ as $k \rightarrow 0$, assuming that $\mathcal{W}(k) \rightarrow 1$ as $k \rightarrow 0$ (as is proper for a volume-normalized window function). The $\psi_n(k)$ obey the orthonormality condition

$$\int_0^\Lambda dk \frac{k^5}{\mathcal{P}(k) \mathcal{W}^2(k)} \psi_n(k) \psi_m(k) = \delta_{mn}, \quad (36)$$

and δ_{mn} is the Kronecker delta. The completeness relation can be written

$$\delta(k - k_0)|_{k \in [0, \Lambda]} = \frac{k_0^5}{\mathcal{P}(k_0) \mathcal{W}^2(k_0)} \sum_n \psi_n(k) \psi_n(k_0), \quad (37)$$

where the range of k is restricted to the compact interval $[0, \Lambda]$.

Although we have given an explicit form for the ψ_n in order to demonstrate existence, the argument does not depend in detail on Eq. (35). The only important properties are Eqs. (36)–(37), which follow from Condition (iv).

3.2. The path integral measure

Since any real, C^0 function \bar{R} obeying the boundary conditions $\bar{R}(\mathbf{k}) \xrightarrow{k \rightarrow 0} 0$ and $\bar{R}(\mathbf{k}) \xrightarrow{k \rightarrow \Lambda} 0$ can be expanded in the form (32), one can formally integrate over all such \bar{R} by integrating over the coefficients $\bar{R}_{\ell n}^m$. This prescription has been widely used for obtaining explicit results from path integral calculations. (For a textbook treatment, see Ref. [95].) In the present case, it must be remembered that one should include in the integral only those $\bar{R}(\mathbf{x})$ which are real, and can therefore correspond to a physical curvature perturbation in the universe. Since the $Y_{\ell m}$ are complex, this means that instead of integrating unrestrictedly over the $\bar{R}_{\ell n}^m$, the reality conditions (33) must be respected. A simple way to achieve this is to integrate only over those $a_{\ell n}^m$ or $b_{\ell n}^m$ with $m \geq 0$. The $m = 0$ modes must be treated specially since the a and b coefficients vanish for odd and even ℓ , respectively.

The integral over real \bar{R} can now be written

$$\int_{\mathbf{R}} [d\bar{R}] = \left[\prod_{\ell=0}^{\infty} \prod_{m=1}^{\infty} \prod_{n=1}^{\ell} \mu \int_{-\infty}^{\infty} da_{\ell n}^m \int_{-\infty}^{\infty} db_{\ell n}^m \right] \left[\prod_{\substack{r=0 \\ r \text{ even}}}^{\infty} \prod_{s=1}^{\infty} \tilde{\mu} \int_{-\infty}^{\infty} da_{r|s}^0 \int_{-\infty}^{\infty} db_{r+1|s}^0 \right], \quad (38)$$

where the subscript \mathbf{R} on the integral indicates schematically that only real $\bar{R}(\mathbf{x})$ are included. The constants μ and $\tilde{\mu}$ account for the Jacobian determinant which arises in writing $\int[d\bar{R}]$ in terms of the harmonic coefficients $\bar{R}_{\ell|n}^m$. Their precise form is of no importance in the present calculation.

As we noticed above, the detailed form of the measure (38) is not absolutely necessary for our argument. The important aspect is that each a or b integral (where $m \geq 0$) can be carried out independently. For this purpose it is sufficient that the spectrum of partial waves is discrete, which follows from the (artificial) compactness of momentum space. However, although it is necessary to adopt some regulator in order to write the path integral measure in a concrete form such as (38), we expect the answer to be independent of the specific regulator which is chosen. In the present context, this means that our final expressions should not depend on Λ , so that the passage to the $\Lambda \rightarrow \infty$ limit becomes trivial.

3.3. The total fluctuation ϵ and the spectrum $\varrho(k)$

There are at least two useful ways in which one might attempt to measure the strength of fluctuations in \bar{R} . The first is the *total smoothed fluctuation* at a given point $\mathbf{x} = \mathbf{x}_0$. By a suitable choice of coordinates, we can always arrange that \mathbf{x}_0 is the origin, so the condition is $\bar{R}(\mathbf{0}) = \epsilon$. When \bar{R} is smoothed on scales of order the horizon size this gives a measure of the fluctuation in each Hubble volume, since distances of less than a horizon size no longer have any meaning. For example, Shibata & Sasaki [96] have proposed that ϵ defined in this way represents a useful criterion for the formation of PBHs, with formation occurring whenever ϵ exceeds a threshold value ϵ_{th} of order unity [97]. This measure of the fluctuation is non-local in momentum space. Making use of the relation $\int d\Omega(\theta, \phi) Y_{\ell m}(\theta, \phi) = \sqrt{4\pi} \delta_{\ell,0} \delta_{m,0}$ for the homogeneous mode of the spherical harmonics, one can characterize ϵ as

$$\int \frac{d^3k}{(2\pi)^3} \bar{R}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}|_{\mathbf{x}=0} = \frac{\sqrt{4\pi}}{(2\pi)^3} \int dk k^2 \sum_{n=1}^{\infty} a_{0|n}^0 \psi_n(k) = \epsilon. \quad (39)$$

On the other hand, one might be interested in contributions to the total smoothed fluctuation in each Hubble volume which arise from features in the spectrum near some characteristic scale of wavenumber k . For this reason, we consider a second possible measure of fluctuations, which we call the *fluctuation spectrum*, and which is defined by the condition $\varrho(k) = d\bar{R}(\mathbf{0})/d \ln k$. (Thus, the total smoothed fluctuation can be obtained by integrating its spectrum according to the usual rule, viz, $\epsilon = \int \varrho(k) d \ln k$.) This condition is local in \mathbf{k} -space. Differentiating (39), one can characterize $\varrho(k)$ as

$$\varrho(k) = \frac{\sqrt{4\pi}}{(2\pi)^3} \sum_{n=1}^{\infty} a_{0|n}^0 k^3 \psi_n(k). \quad (40)$$

This is a functional constraint.

We will calculate the statistics of both the total fluctuation ϵ and its spectrum $\varrho(k)$. In each case, the calculation is easily adapted to other observables which are non-local

or local in momentum space, respectively. Indeed, both the non-local ϵ and the local $\varrho(k)$ are members of a large class of observables, which we can collectively denote ζ , and which all share nearly-Gaussian statistics. Specifically, Eqs. (39) and (40) can be written in a unified manner in the form

$$\sum_{n=1}^{\infty} a_{0|n}^0 \Sigma_n(k) = \frac{(2\pi)^3}{\sqrt{4\pi}} \zeta(k), \quad (41)$$

where the Σ_n are defined by

$$\Sigma_n = \begin{cases} \int_0^\Lambda dk k^2 \psi_n(k) & \text{total fluctuation, } \zeta = \epsilon; \\ k^3 \psi_n(k) & \text{fluctuation spectrum, } \zeta = \varrho(k). \end{cases} \quad (42)$$

Note that in the case of ϵ , the Σ_n are independent of k . Any characteristic which can be put in this form, with a coupling only to the real zero-modes $a_{0|n}^0$ of $\bar{\mathbf{R}}$, will necessarily develop nearly-Gaussian (*i.e.*, weakly non-Gaussian) statistics. More general choices of characteristic are possible, which cannot be cast in the form (41). For example, one can consider characteristics which depend non-linearly on the $a_{0|n}^0$. Such characteristics will generally lead to strongly non-Gaussian probabilities. The Gaussianity of the final probability distribution depends on the geometry of the constraint surface in an analogous way to the decoupling of the Fadeev-Popov ghost fields in gauge field theory [98]. These non-Gaussian choices of characteristic can also be handled by generalizing our technique, but we do not consider them here.

4. The probability density function for ϵ

We first calculate the probability density for the non-local constraint ϵ , given by Eq. (39). The expression is

$$\mathbf{P}(\epsilon) \propto \int_{\mathbf{R}} [d\bar{\mathbf{R}}] \mathbf{P}[\bar{\mathbf{R}}] \delta \left[\sum_{n=1}^{\infty} a_{0|n}^0 \Sigma_n - \frac{(2\pi)^3}{\sqrt{4\pi}} \epsilon \right]. \quad (43)$$

To obtain this density, one treats ϵ as a collective coordinate parametrizing part of $\bar{\mathbf{R}}$. The remaining degrees of freedom, which are orthogonal to ϵ , are denoted $\bar{\mathbf{R}}^\perp$. Therefore the functional measure can be broken into $[d\bar{\mathbf{R}}] \propto [d\bar{\mathbf{R}}^\perp] d\epsilon$. After integrating the functional density $\mathbf{P}[\bar{\mathbf{R}}] [d\bar{\mathbf{R}}]$ over $\bar{\mathbf{R}}^\perp$, the quantity which is left is the probability density $\mathbf{P}(\epsilon) d\epsilon$. In this case, the integration over the orthogonal degrees of freedom $\bar{\mathbf{R}}^\perp$ is accomplished via the δ -function, which filters out only those members of the ensemble which satisfy Eq (39). We emphasize that this is a conventional δ -function, not a δ -functional. There is no need to take account of a Fadeev-Popov type factor because the Jacobian associated with the constraint (41) is field-independent, in virtue of the linearity of (39) in $a_{0|n}^0$.

4.1. The Gaussian case

We first give the calculation in the approximation that only the two-point function is retained. In this approximation, the probability distribution of ϵ will turn out to

be purely Gaussian, which allows us to develop our method without the distractions introduced by including non-Gaussian effects.

If all correlation functions of order three and higher are set to zero, then $\mathbf{P}[\bar{\mathbf{R}}] \propto \Gamma[\bar{\mathbf{R}}]$. Using (5), one can write

$$\Gamma[\bar{\mathbf{R}}] = \exp \left(-\frac{1}{2} \int d\Omega \int k^2 dk \frac{k^3}{(2\pi)^3 2\pi^2} \frac{1}{\mathcal{P}(k) \mathcal{W}^2(k)} \right. \\ \left. \times \sum_{\ell_1, m_1, n_1} \sum_{\ell_2, m_2, n_2} \bar{R}_{\ell_1|n_1}^{m_1} \bar{R}_{\ell_2|n_2}^{m_2\dagger} Y_{\ell_1 m_1}(\theta, \phi) Y_{\ell_2 m_2}^\dagger(\theta, \phi) \psi_{n_1}(k) \psi_{n_2}(k) \right). \quad (44)$$

The harmonics $Y_{\ell m}$ and ψ_n integrate out of this expression entirely, using the orthonormality relation (36) and the spherical harmonic completeness relation $\int d\Omega Y_{\ell_1 m_1} Y_{\ell_2 m_2}^\dagger = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}$. Moreover, after rewriting the a and b coefficients with $m < 0$ in terms of the $m > 0$ coefficients, we obtain

$$\Gamma[\bar{\mathbf{R}}] = \exp \left(-\frac{1}{2\pi^2 (2\pi)^3} \sum_{\ell=0}^{\infty} \sum_{m=1}^{\ell} \sum_{n=1}^{\infty} |a_{\ell|n}^m|^2 + |b_{\ell|n}^m|^2 - \frac{1}{4\pi^2 (2\pi)^3} \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{\infty} \sum_{n=1}^{\infty} |a_{\ell|n}^0|^2 + |b_{\ell+1|n}^0|^2 \right). \quad (45)$$

The δ -function in (43) constrains one of the $a_{0|n}^0$ in terms of ϵ and the other coefficients. It is possible to evaluate $\mathbf{P}(\epsilon)$ by integrating out the δ -function immediately. This would involve solving the constraint for $a_{0|0}^0$ (for example) and replacing it in the integrand with its expression in terms of the other $a_{0|n}^0$ and ϵ . However, this does not turn out to be a convenient procedure, for the same reasons that one encounters when gauge-fixing in field theory. Instead, we introduce the Fourier representation of the δ -function,

$$\mathbf{P}(\epsilon) \propto \int_{\mathbf{R}} [d\bar{\mathbf{R}}] \int_{-\infty}^{\infty} dz \Gamma[\bar{\mathbf{R}}] \exp \left(iz \left[\sum_{n=1}^{\infty} a_{0|n}^0 \Sigma_n - \frac{(2\pi)^3}{\sqrt{4\pi}} \epsilon \right] \right), \quad (46)$$

where the functional measure is understood to be Eq. (38). The final answer is obtained by integrating out z together with all of the a and b coefficients. In order to achieve this, it is necessary to decouple $a_{0|n}^0$, z and ϵ from each other by successively completing the square in $a_{0|0}^0$ and z . Working with $a_{0|0}^0$ first, we find

$$\exp \left(-\frac{1}{4\pi^2} \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} |a_{0|n}^0|^2 + iz \sum_{n=1}^{\infty} a_{0|n}^0 \Sigma_n \right) \\ = \exp \left(-\frac{1}{4\pi^2} \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} (a_{0|n}^0 - 2\pi^2 (2\pi)^3 iz \Sigma_n)^2 - (2\pi)^3 \pi^2 z^2 \Sigma^2 \right), \quad (47)$$

where we have introduced a function Σ^2 , defined by $\Sigma^2 = \sum_{n=1}^{\infty} \Sigma_n^2$. In the final probability distribution, Σ^2 will turn out to be the variance of ϵ . From Eq. (47), it is clear that making the transformation $a_{0|n}^0 \mapsto a_{0|n}^0 + 2\pi^2 (2\pi)^3 iz \Sigma_n$ suffices to decouple $a_{0|n}^0$ from z . The measure, Eq. (38), is formally invariant under this transformation. Exactly the same procedure can now be applied to z and ϵ , giving

$$\exp \left(-(2\pi)^3 \pi^2 z^2 \Sigma^2 - \frac{(2\pi)^3}{\sqrt{4\pi}} i\epsilon z \right) = \exp \left[-(2\pi)^3 \pi^2 \Sigma^2 \left(z + \frac{i\epsilon}{2\pi^2 \sqrt{4\pi} \Sigma^2} \right)^2 - \frac{\epsilon^2}{2\Sigma^2} \right]. \quad (48)$$

As before, the finite shift $z \mapsto z - i\epsilon/2\pi^2\sqrt{4\pi}\Sigma^2$ leaves the measure intact and decouples z and ϵ . The a , b and z integrals can be done independently, but since they do not involve ϵ they contribute only an irrelevant normalization to $\mathbf{P}(\epsilon)$. Thus, we obtain Gaussian statistics for ϵ ,

$$\mathbf{P}(\epsilon) \propto \exp\left(-\frac{\epsilon^2}{2\Sigma^2}\right). \quad (49)$$

It remains to evaluate the variance Σ^2 . In the present case, we have $\Sigma_n = \int_0^\Lambda dk k^2 \psi_n(k)$. From the completeness relation Eq. (37), it follows that

$$\sum_n k_0^2 \psi_n(k_0) k^2 \psi_n(k) = \frac{k^2 \mathcal{P}(k_0) \mathcal{W}^2(k_0)}{k_0^3} \delta(k - k_0). \quad (50)$$

Σ^2 is now obtained by integrating term-by-term under the summation. The result coincides with the *smoothed* conventional variance (compare Eq. (4)),

$$\Sigma_\Lambda^2(k_H) = \int_0^\Lambda d \ln k \mathcal{W}^2(k; k_H) \mathcal{P}(k). \quad (51)$$

Thus, as expected, Eq. (49) reproduces the Gaussian distribution (3) which was derived on the basis of the central limit theorem, with the proviso that the parameters (such as Σ^2) describing the distribution of ϵ are associated with the smoothed field $\bar{\mathbf{R}}$ rather than the microphysical field \mathbf{R} . Note that Σ^2 is therefore implicitly a function of scale, with the scale dependence entering through the window function.

In particular, it was only necessary to use the completeness relation to obtain this result, which follows from Condition (iv) in Section 3.1.

4.2. The non-Gaussian case

The non-Gaussian case is a reasonably straightforward extension of the calculation described in the preceding section, with the term $\Upsilon^{(0)}$ in Eq. (26) (which was dropped in the previous section) included. However, some calculations become algebraically long, and there are subtleties connected to the appearance of the bispectrum.

The inclusion of $\Upsilon^{(0)}$ corrects the pure Gaussian statistics by a quantity proportional to the three-point function, $\langle \mathcal{R}\mathcal{R}\mathcal{R} \rangle$, which is given in Eq. (6). This correction is written in terms of the representative spectrum $\bar{\mathcal{P}}^2$, which describes when the slow-roll prefactor, given by the amplitude of the spectrum, should be evaluated [40]. For modes which cross the horizon almost simultaneously, $k_1 \sim k_2 \sim k_3$, this prefactor should be $\bar{\mathcal{P}}^2 = \mathcal{P}(k)^2$, where k is the common magnitude of the k_i . In the alternative case where one \mathbf{k} -mode crosses appreciably before the other two, $\bar{\mathcal{P}}^2$ should be roughly given by

$$\bar{\mathcal{P}}^2 = \mathcal{P}(\max k_i) \mathcal{P}(\min k_i). \quad (52)$$

Since the difference between this expression and $\mathcal{P}(k)^2$ when all k are of the same magnitude is very small, it is reasonable to adopt Eq. (52) as our definition of $\bar{\mathcal{P}}^2$. We stress that this prescription relies on the conservation of \mathcal{R} outside the horizon [89], and therefore would become more complicated if extended to a multiple field scenario.

With this parametrization, the probability measure on the ensemble is obtained by combining (5), (26), (27) and (6),

$$\mathbf{P}[\bar{\mathbf{R}}] \propto \Gamma[\bar{\mathbf{R}}] \left(1 - \frac{1}{6} \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{(2\pi)^6 2\pi^2} \delta(\Delta) \frac{\bar{\mathcal{P}}^2 \mathcal{A}}{\prod_i \mathcal{P}(k_i)} \frac{\bar{\mathbf{R}}(\mathbf{k}_1) \bar{\mathbf{R}}(\mathbf{k}_2) \bar{\mathbf{R}}(\mathbf{k}_3)}{\mathcal{W}(k_1) \mathcal{W}(k_2) \mathcal{W}(k_3)} \right). \quad (53)$$

This expression should be integrated with the constraint (39) and measure (38) to obtain the probability $\mathbf{P}(\epsilon)$. At first this appears to lead to an undesirable consequence, since the integral of any odd function of $\bar{\mathbf{R}}$ against $\Gamma[\bar{\mathbf{R}}]$ must be zero. It may therefore seem as if the non-Gaussian corrections we are trying to obtain will evaluate to zero. This would certainly be correct if the integral were unconstrained. However, the presence of the constraint δ -function means that the shifts of $a_{0|n}^0$ and z which are necessary to decouple the integration variables give rise to a non-vanishing correction.

The finite shift necessary to decouple $a_{0|n}^0$ and z is not changed by the presence of non-Gaussian corrections, since it only depends on the argument of the exponential term. This is the same in the Gaussian and non-Gaussian cases. After making this shift, which again leaves the measure invariant, the integration becomes

$$\mathbf{P}(\epsilon) \propto \int_{\mathbf{R}} [d\bar{\mathbf{R}}] \int_{-\infty}^{\infty} dz \Gamma[\bar{\mathbf{R}}] \exp \left(-(2\pi)^3 \pi^2 \Sigma^2 z^2 - \frac{(2\pi)^3}{\sqrt{4\pi}} i z \epsilon \right) (1 - J_0 - J_2), \quad (54)$$

where J_0 is given by

$$\int d^3 k_1 d^3 k_2 d^3 k_3 \frac{2\pi^4 (2\pi)^3}{3(4\pi)^{3/2}} \delta(\Delta) \frac{\bar{\mathcal{P}}^2 \mathcal{A}}{\prod_i \mathcal{P}(k_i)} \sum_{n_1, n_2, n_3} i^3 z^3 \Sigma_{n_1} \Sigma_{n_2} \Sigma_{n_3} \frac{\psi_{n_1}(k_1) \psi_{n_2}(k_2) \psi_{n_3}(k_3)}{\mathcal{W}(k_1) \mathcal{W}(k_2) \mathcal{W}(k_3)},$$

and J_2 is

$$\begin{aligned} & \left[\int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{6(2\pi)^3 \sqrt{4\pi}} \delta(\Delta) \frac{\bar{\mathcal{P}}^2 \mathcal{A}}{\prod_i \mathcal{P}(k_i)} \sum_{n_1} \sum_{\ell_2, m_2, n_2} \sum_{\ell_3, m_3, n_3} \right. \\ & \quad \times i z \Sigma_{n_1} \frac{\psi_{n_1}(k_1)}{\mathcal{W}(k_1)} \bar{\mathbf{R}}_{\ell_2|n_2}^{m_2} \bar{\mathbf{R}}_{\ell_3|n_3}^{m_3} Y_{\ell_2 m_2}(\theta_2, \phi_2) Y_{\ell_3 m_3}(\theta_3, \phi_3) \frac{\psi_{n_2}(k_2) \psi_{n_3}(k_3)}{\mathcal{W}(k_2) \mathcal{W}(k_3)} \left. \right] \\ & + [\mathbf{1} \rightleftharpoons \mathbf{2}] + [\mathbf{1} \rightleftharpoons \mathbf{3}]. \end{aligned} \quad (55)$$

The symbol $[\mathbf{1} \rightleftharpoons \mathbf{2}]$ represents the expression in square brackets $[\dots]$ with the label 1 exchanged with the label 2, and similarly for $[\mathbf{1} \rightleftharpoons \mathbf{3}]$. The range of the m_2, m_3 summations is from $-\ell_2$ to ℓ_2 and $-\ell_3$ to ℓ_3 , respectively. In addition, the shift of $a_{0|n}^0$ generates other terms linear and cubic in the $\bar{\mathbf{R}}_{\ell|n}^m$, but these terms do not contribute to $\mathbf{P}(\epsilon)$ and we have omitted them from (54).

After shifting z to decouple z and ϵ , the integrals J_0 and J_2 develop terms proportional to z^0 , z , z^2 and z^3 . Of these, only the z^0 and z^2 survive the final z integration. Correspondingly, we suppress terms linear and cubic in z from the following expressions. The integral J_0 becomes

$$\begin{aligned} J_0 = & \int d^3 k_1 d^3 k_2 d^3 k_3 \frac{\pi^2 (2\pi)^3}{3(4\pi)^2} \left(\frac{1}{16\pi^5} \frac{\epsilon^3}{\Sigma^6} - 3 \frac{z^2 \epsilon}{\Sigma^2} \right) \delta(\Delta) \frac{\bar{\mathcal{P}}^2 \mathcal{A}}{\prod_i \mathcal{P}(k_i)} \\ & \times \sum_{n_1, n_2, n_3} \Sigma_{n_1} \Sigma_{n_2} \Sigma_{n_3} \frac{\psi_{n_1}(k_1) \psi_{n_2}(k_2) \psi_{n_3}(k_3)}{\mathcal{W}(k_1) \mathcal{W}(k_2) \mathcal{W}(k_3)}, \end{aligned} \quad (56)$$

whereas J_2 simplifies to

$$J_2 = \left[\int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{48\pi^3 (2\pi)^3} \frac{\epsilon}{\Sigma^2} \sum_{n_1} \sum_{\ell_2, m_2, n_2} \sum_{\ell_3, m_3, n_3} \right. \\ \left. \times \Sigma_{n_1} \frac{\psi_{n_1}(k_1)}{\mathcal{W}(k_1)} \bar{R}_{\ell_2|n_2}^{m_2} \bar{R}_{\ell_3|n_3}^{m_3} Y_{\ell_2 m_2}(\theta_2, \phi_2) Y_{\ell_3 m_3}(\theta_3, \phi_3) \frac{\psi_{n_2}(k_2) \psi_{n_3}(k_3)}{\mathcal{W}(k_2) \mathcal{W}(k_3)} \right] \\ + [\mathbf{1} \rightleftharpoons \mathbf{2}] + [\mathbf{1} \rightleftharpoons \mathbf{3}], \quad (57)$$

and the m summations are still over the entire range, $-\ell_2 \leq m_2 \leq \ell_2$ (and similarly for m_3). Thus J_0 contains corrections proportional to ϵ and ϵ^3 , whereas J_2 only contains corrections proportional to ϵ .

The a , b and z integrations can now be performed, after the integrand has been written entirely in terms of the $a_{\ell|n}^m$ and $b_{\ell|n}^m$ with $m \geq 0$. There are no a or b integrations in J_0 . In J_2 , there are no z integrations, but the a and b integrations involved in the product $\bar{R}_{\ell_2|n_2}^{m_2} \bar{R}_{\ell_3|n_3}^{m_3}$ fix $\ell_2 = \ell_3$, $m_2 = m_3$ and $n_2 = n_3$. One then uses the spherical harmonic completeness relation,

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta_1, \phi_1) Y_{\ell m}^{\dagger}(\theta_2, \phi_2) = \delta(\phi_1 - \phi_2) \delta(\cos \theta_1 - \cos \theta_2) \quad (58)$$

and the equivalent relationship for the ψ -harmonics, Eq. (37), to obtain

$$J_2 = \left[\int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{24\pi} \frac{\epsilon}{\Sigma^2} \delta(\Delta) \frac{\bar{\mathcal{P}}^2 \mathcal{A}}{\prod_i \mathcal{P}(k_i) \mathcal{W}(k_i)} \frac{\mathcal{P}(k_2) \mathcal{W}^2(k_2)}{k_2^3} \sum_n \Sigma_n \psi_n(k_1) \delta(\mathbf{k}_2 + \mathbf{k}_3) \right] \\ + [\mathbf{1} \rightleftharpoons \mathbf{2}] + [\mathbf{1} \rightleftharpoons \mathbf{3}]. \quad (59)$$

The terms with 1 exchanged with 2 and 3 do not generate quantitatively different integrands and can be absorbed into an overall factor of 3.

J_0 involves only z integrations. It can be written

$$J_0 = \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{96\pi^2} \left(\frac{\epsilon^3}{\Sigma^6} - 3 \frac{\epsilon}{\Sigma^4} \right) \delta(\Delta) \frac{\bar{\mathcal{P}}^2 \mathcal{A}}{\prod_i \mathcal{P}(k_i) \mathcal{W}(k_i)} \\ \times \sum_{n_1, n_2, n_3} \Sigma_{n_1} \Sigma_{n_2} \Sigma_{n_3} \psi_{n_1}(k_1) \psi_{n_2}(k_2) \psi_{n_3}(k_3). \quad (60)$$

To simplify these expressions further, it is necessary to obtain the value of the sum $\sum_{n=1}^{\infty} \Sigma_n \psi_n(k)$.

Reasoning as before from the completeness relation Eq. (37), it follows that

$$\sum_{n=1}^{\infty} \Sigma_n \psi_n(k) = \frac{\mathcal{P}(k) \mathcal{W}^2(k)}{k^3}. \quad (61)$$

From this, it is straightforward to show that J_0 behaves like

$$J_0 = \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{96\pi^2 \prod_i k_i^3 \mathcal{W}^{-1}(k_i)} \delta(\Delta) \bar{\mathcal{P}}^2 \mathcal{A} \left(\frac{\epsilon^3}{\Sigma^6} - 3 \frac{\epsilon}{\Sigma^4} \right), \quad (62)$$

where Σ^2 is the smoothed variance, Eq. (51). On the other hand J_2 becomes

$$J_2 = \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{24\pi/3} \frac{\epsilon}{\Sigma^2} \delta(\Delta) \frac{\bar{\mathcal{P}}^2}{\mathcal{P}(k_2)} \mathcal{W}(k_1) \mathcal{A} \frac{\delta(\mathbf{k}_2 + \mathbf{k}_3)}{k_1^3 k_2^3}. \quad (63)$$

After integrating out \mathbf{k}_3 and the angular part of \mathbf{k}_1 and \mathbf{k}_2 , this is the same as

$$J_2 = 2\pi \int dk_2 k_2^2 \int dk_1 \delta(k_1) \frac{\epsilon}{\Sigma^2} \mathcal{W}(k_1) \frac{1}{k_2^3 \mathcal{P}(k_2)} \lim_{k_1 \rightarrow 0} \mathcal{A} \frac{\mathcal{P}(k_1)}{k_1^3}, \quad (64)$$

where we have used the fact that k_1 is constrained to zero by the δ -function to evaluate the bispectrum \mathcal{A} in the ‘squeezed’ limit where one of the momenta goes to zero [40, 89, 88]. In this limit, $\min k_i = k_1$ and $\max k_i = k_2 = k_3$, so it is possible to expand $\bar{\mathcal{P}}^2$ unambiguously. Moreover, $\lim_{k_1 \rightarrow 0} \mathcal{A} = \alpha k_2^3$ is proportional to k_2^3 , so $J_2 = 0$ if $\mathcal{P}(k)/k^3 \rightarrow 0$ as $k \rightarrow 0$. This is the sharper condition on how strongly large-scale power is suppressed which was anticipated in Section 3.1. It requires that $\mathcal{P}(k)$ cuts off on long lengthscales faster than k^3 . If this does not occur, then the integral diverges. (There is a marginal case where $\mathcal{P}(k)/k^3$ tends to a finite limit as k approaches zero. We assume that this case is not physically relevant.)

The J_2 integral contains a $\delta(\mathbf{k}_2 + \mathbf{k}_3)$. It can therefore be interpreted as counting contributions to the bispectrum which come from a correlation between the modes \mathbf{k}_2 and \mathbf{k}_3 , in a background created by \mathbf{k}_1 , which exited the horizon in the asymptotic past. As we have already argued, modes of this sort are included in the FRW background around which we perturbed to obtain the correlation functions of \mathcal{R} , so we can anticipate that its contribution should be zero, as the above analysis shows explicitly. In this interpretation, the condition $\mathcal{P}(k)/k^3 \rightarrow 0$ as $k \rightarrow 0$ is the condition that the perturbation does not destroy the FRW background. Indeed, fluctuations on very large scales in effect describe transitions from one FRW world to another via a shift in the zero-momentum modes of the background metric. In this case, there is only one such mode, which is the scale factor $a(t)$. These transitions are rather like changing the vacuum state in a quantum field theory. As a result, fluctuations of a large volume of the universe between one FRW state and another are strongly suppressed.

For the case of fluctuations on the scale of a Hubble volume, therefore, the probability distribution should be written

$$\mathbf{P}(\epsilon) = \frac{1}{\sqrt{2\pi}\Sigma} \left[1 - \left(\frac{\epsilon^3}{\Sigma^6} - 3 \frac{\epsilon}{\Sigma^4} \right) J \right] \exp \left(-\frac{\epsilon^2}{2\Sigma^2} \right), \quad (65)$$

where we have used the fact that the corrections are odd in ϵ , and therefore do not contribute to the overall normalization of $\mathbf{P}(\epsilon)$. The (dimensionless) coefficient J is

$$J = \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{96\pi^2 \prod_i k_i^3 \mathcal{W}^{-1}(k_i)} \delta(\Delta) \bar{\mathcal{P}}^2 \mathcal{A}. \quad (66)$$

This expression is remarkably simple. Indeed, although the explicit expression (66) is preferable for calculation, it can be recast directly as the integrated bispectrum with respect to \mathcal{W} :

$$J = \frac{1}{48(2\pi)^3(2\pi^2)^3} \int d^3 k_1 d^3 k_2 d^3 k_3 \langle \mathcal{R}(\mathbf{k}_1) \mathcal{R}(\mathbf{k}_2) \mathcal{R}(\mathbf{k}_3) \rangle \mathcal{W}(k_1) \mathcal{W}(k_2) \mathcal{W}(k_3) \quad (67)$$

As a consistency check, we note that the expectation of ϵ , defined by $\mathbf{E}(\epsilon) = \int \epsilon \mathbf{P}(\epsilon) d\epsilon$ satisfies $\mathbf{E}(\epsilon) = 0$. This is certainly necessary, since the universe must contain as many underdense regions as overdense ones, but is a non-trivial restriction, since

both the ϵ and ϵ^3 corrections to $\mathbf{P}(\epsilon)$ do not separately average to zero. The particular combination of coefficients in (65) is the unique correction [up to $\mathcal{O}(\epsilon^3)$, containing only odd powers of ϵ] which maintains $\mathbf{E}(\epsilon) = 0$.

Finally, we note that Eqs. (65)–(66) do not explicitly involve the cut-off Λ , except as a limit of integration and in quantities such as Σ_Λ^2 and \mathcal{W}_Λ which possess a well-defined, finite limit at large Λ . As a result, there is no obstruction to taking the $\Lambda \rightarrow \infty$ limit to remove the regulator entirely.

4.3. When is perturbation theory valid?

It is known from explicit calculation that the bispectrum is of order \mathcal{P}^2 multiplied by a small quantity, f_{NL} , which is predicted to be small when slow-roll is valid. It is therefore reasonable to suppose that whenever the window functions \mathcal{W} are peaked around some probe wavenumber k_* , one has the approximate relations (on order of magnitude) $\Sigma^2 \sim \mathcal{P}_*$ and $J \sim \mathcal{P}_*^2$, where \mathcal{P}_* represents the spectrum evaluated at $k = k_*$. Since the ϵ^3 correction dominates for $\epsilon > \sqrt{3}\Sigma$, this means that for ϵ not too large, $\epsilon \ll \mathcal{P}_*^{-3/2}$, the perturbative correction we have calculated will be small. As ϵ increases, so that $\epsilon \gg \mathcal{P}_*^{-3/2}$, perturbation theory breaks down and the power series in ϵ needs resummation. In any case, at such high values of ϵ , the calculation described above ought to be supplemented by new physics which can be expected to become important at high energy density. The details of these corrections presumably do not matter too much, because at any finite order, the fast-decaying exponential piece suppresses any contributions from large values of ϵ .

At some value of ϵ , corrections coming from the trispectrum can be expected to become comparable to those coming from the bispectrum that we have computed. Let us assume that for intermediate values of ϵ , such that $\Sigma \ll \epsilon \ll \mathcal{P}_*^{-3/2}$, a contribution of the form $\epsilon^4 \Sigma^{-8} J_{\text{tri}}$ dominates the correction from the trispectrum, where J_{tri} is the integrated trispectrum with respect to \mathcal{W} . The trispectrum contribution will be subdominant provided

$$\frac{\epsilon^3}{\Sigma^6} J \gtrsim \frac{\epsilon^4}{\Sigma^8} J_{\text{tri}}, \quad (68)$$

which is true whenever $\epsilon < \Sigma^2 J / J_{\text{tri}}$. The trispectrum from slow-roll inflation has not yet been explicitly computed, but it is expected to be proportional to three powers of \mathcal{P} . Therefore, $\Sigma^2 J \sim \mathcal{P}_*^3$ and $J_{\text{tri}} \sim \mathcal{P}_*^3$ can be expected to be of roughly equal orders of magnitude, up to a numerical coefficient which can be estimated to be $J/J_{\text{tri}} \simeq 16\pi \sim 50$. (This number comes from $4!/3! = 4$ and a factor 4π from the extra angular integrations which arise when integrating the trispectrum.) Thus for values of ϵ of order unity, it is reasonable to expect the correction from the bispectrum to dominate the correction from the trispectrum and higher correlation functions. However, we caution that if the trispectrum is anomalously large, or for much larger values of ϵ , higher-order terms in the perturbation series will become relevant.

5. The probability density function for $\varrho(k)$

The probability density function for $\varrho(k)$ can be obtained by a reasonably straightforward modification of the above argument, taking account of the fact that the constraint, Eq. (40), is now a functional constraint. This means that when splitting the functional measure $[d\bar{\mathbf{R}}]$ into a product of $[d\varrho(k)]$ and the orthogonal degrees of freedom $[d\bar{\mathbf{R}}^\perp]$, the result after integrating out the $\bar{\mathbf{R}}^\perp$ coordinates gives a functional probability density in $[d\varrho(k)]$. In particular, the δ -function is now represented as

$$\int [dz] \exp \left[i \int dk z(k) \left(\sum_{n=1}^{\infty} a_{0|n}^0 k^3 \psi_n(k) - \frac{(2\pi)^3}{\sqrt{4\pi}} \varrho(k) \right) \right]. \quad (69)$$

In order to carry out this calculation, we write $z(k)$ formally as

$$z(k) = \sum_{n=1}^{\infty} \frac{k^2}{\mathcal{P}(k) \mathcal{W}^2(k)} z_n \psi_n(k). \quad (70)$$

The integration measure $\int [dz]$ becomes $\prod_n \check{\mu} \int_{-\infty}^{\infty} dz_n$, where, as before, $\check{\mu}$ is a field-independent Jacobian representing the change of variables from $z(k) \mapsto z_n$. Its value is not relevant to the present calculation. In addition, we introduce a set of coefficients $\tilde{\varrho}_n$ to describe $\varrho(k)$,

$$\frac{\varrho(k)}{k^3} = \sum_{n=1}^{\infty} \tilde{\varrho}_n \psi_n(k). \quad (71)$$

The $\tilde{\varrho}_n$ can be calculated using the rule $\tilde{\varrho}_n = \int_0^\Lambda dk k^2 \mathcal{P}^{-1}(k) \mathcal{W}^{-2}(k) \varrho(k) \psi_n(k)$. Note that in order to do so, we have made the implicit assumption that $\varrho(k)/k^3 \rightarrow 0$ as $k \rightarrow 0$, in order that (71) is compatible with the boundary conditions of the $\psi_n(k)$. We again see the suppression of power in modes with low k .

With these choices, the constraint δ -function becomes

$$\prod_n \check{\mu} \int_{-\infty}^{\infty} dz_n \exp \left[i \sum_{m=1}^{\infty} \left(a_{0|m}^0 z_m - \frac{(2\pi)^3}{\sqrt{4\pi}} z_m \tilde{\varrho}_m \right) \right], \quad (72)$$

As opposed to the nonlocal case of ϵ , where a single extra integration over z coupled to ϵ , we now have a situation where a countably infinite tower of integrations over z_n couple to the coefficients $\tilde{\varrho}_n$. In all other respects, however, this calculation is now much the same as the nonlocal one, and can be carried out in the same way. The shift of variables necessary to decouple $a_{0|n}^0$ and z_n is

$$a_{0|n}^0 \mapsto a_{0|n}^0 + i2\pi^2(2\pi)^3 z_n; \quad (73)$$

and the shift necessary to decouple the z_n and $\tilde{\varrho}_n$ is

$$z_n \mapsto z_n - \frac{i\tilde{\varrho}_n}{2\pi^2\sqrt{4\pi}}. \quad (74)$$

When only the two-point function is included, we obtain a Gaussian in the $\tilde{\varrho}_n$,

$$\mathbf{P}[\varrho(k)] \propto \exp \left(-\frac{1}{2} \sum_n \tilde{\varrho}_n^2 \right). \quad (75)$$

The sum over the $\tilde{\varrho}_n$ can be carried out using the completeness and orthogonality relations for the $\psi_n(k)$, and Eq. (71),

$$\sum_n \tilde{\varrho}_n^2 = \int d \ln k \frac{\varrho^2(k)}{\mathcal{P}(k)\mathcal{W}^2(k)}. \quad (76)$$

Using this expression, and integrating over all $\varrho(k)$ which give rise to a fluctuation of strength ϵ , one recovers the Gaussian probability profile Eq. (49) with variance given by Eq. (51). This serves as a consistency check of (75) and (49).

When the non-Gaussian correction $\Upsilon^{(0)}$ is included, one again generates a probability density of the form

$$\mathbf{P}[\varrho(k)] \propto (1 - K_0 - K_2) \exp \left(-\frac{1}{2} \sum_n \tilde{\varrho}_n^2 \right), \quad (77)$$

where K_2 is of the same form as J_2 , and therefore vanishes for the same reasons; and K_0 has the form

$$K_0 = \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{96\pi^2 \prod_i \mathcal{P}(k_i)\mathcal{W}(k_i)} \delta(\Delta) \bar{\mathcal{P}}^2 \left(3 \frac{\varrho(k_1)}{k_1^3} \frac{\mathcal{P}(k_2)\mathcal{W}^2(k_2)}{k_2^5} \delta(\mathbf{k}_2 + \mathbf{k}_3) - \prod_i \frac{\varrho(k_i)}{k_i^3} \right). \quad (78)$$

The first term contains a δ -function which squeezes k_1 into the asymptotic past. It formally vanishes in virtue of our assumption about the behaviour of $\varrho(k)$ near $k = 0$, which is implicit in Eq. (71). As a result, the total probability density for the fluctuation spectrum can be written

$$\mathbf{P}[\varrho(k)] \propto (1 - K) \exp \left(-\frac{1}{2} \int d \ln k \frac{\varrho(k)^2}{\mathcal{P}(k)\mathcal{W}^2(k)} \right), \quad (79)$$

where K is given by

$$K = - \int \frac{d^3 k_1 d^3 k_2 d^3 k_3}{96\pi^2 \prod_i \mathcal{P}(k_i)\mathcal{W}(k_i)} \delta(\Delta) \bar{\mathcal{P}}^2 \prod_i \frac{\varrho(k_i)}{k_i^3}. \quad (80)$$

As before, one can show that this expression is consistent with Eqs. (65)–(66) by integrating over all $\varrho(k)$ which reproduce a total fluctuation of size ϵ , after dropping another term which is squeezed into the asymptotic past owing to the presence of a δ -function. This is a non-trivial consistency check of (79)–(80).

As in the local case, Eqs. (79)–(80) are entirely independent of Λ (except as a limit of integration), so the regulator can be freely removed by setting $\Lambda = \infty$.

6. Conclusions

In this paper we have obtained the connexion between the n -point correlation functions of the primordial curvature perturbation, evaluated at some time t , such as $\langle \mathcal{R}(\mathbf{k}_1) \cdots \mathcal{R}(\mathbf{k}_n) \rangle$, and the probability distribution of fluctuations in the spatial configuration of \mathcal{R} . We have obtained an explicit expression for the probability of a fluctuation of “size” ϵ when \mathcal{R} is smoothed over regions of order the horizon size. This is a probability density in the conventional sense. In addition, we have obtained

an expression for the probability that ϵ has a spectrum of the form $\varrho(k)$, that is, $\int d \ln k \varrho(k) = \epsilon$. (The mapping $\varrho(k) \mapsto \epsilon$ is many-to-one.) This is a functional probability density, and can potentially be used to identify features in the fluctuation spectrum near some specific scale, say of wavenumber $k \simeq k_*$. Our result is independent of statistical reasoning based on the central limit theorem and provides a direct route to incorporate non-Gaussian information from the vertices of the effective quantum field theory of the inflaton into theories of structure formation.

Both these probabilities are Gaussian in the limit where \mathcal{R} only possesses a two-point connected correlation function. If there are higher-order connected correlation functions, then \mathcal{R} exhibits deviations from Gaussian statistics, which we have explicitly calculated using recent determinations of the inflationary three-point function during an epoch of slow-roll inflation. Our method can be extended to incorporate corrections from higher connected n -point functions to any finite order in n . We have not computed these higher corrections, since we anticipate that their contribution is subdominant to the three-point correction (which is already small), and in any case the relevant 4- and higher n -point correlation functions are not yet known explicitly.

Our argument is based on a formal decomposition of the spatial configuration of the curvature perturbation in \mathbf{k} -space into spherical harmonics, together with harmonics along the radial k direction. However, we have emphasized that our results do not depend on the details of this construction, but require only a minimal set of assumptions or conditions. These assumptions are: (A) that the power spectrum $\mathcal{P}(k)$ goes to zero sufficiently fast on large scales, specifically so that $\mathcal{P}(k)/k^3 \rightarrow 0$ as $k \rightarrow 0$. (In addition, in the case of the fluctuation spectrum, we also require $\varrho(k)/k^3 \rightarrow 0$ as $k \rightarrow 0$.) Such a condition is certainly consistent with our understanding of large scale structure in the universe, and within the perturbative approach we are using, we have argued that in fact it describes a self-consistency condition which prevents perturbative fluctuations from destroying the background FRW spacetime. In addition, we require a second condition (B) that the spatial configuration \mathbf{R} can be smoothed $\bar{\mathbf{R}}$ via a window function \mathcal{W} to obtain a configuration for which $\bar{\mathbf{R}} \rightarrow 0$ as $k \rightarrow \infty$, for which it is fair to compare $\bar{\mathbf{R}}$ to the primordial power spectrum.

In addition to these fundamental assumptions, which relate to the behaviour of real physical quantities, a large part of the calculation relied on an auxiliary technical construction. This construction is based on an artificial compactification of momentum space, implemented by a hard cutoff Λ . There is an associated boundary condition on $\bar{\mathbf{R}}$ at $k = \Lambda$ which discretizes the harmonics (partial waves) in k . However, in both the non-local (total fluctuation, ϵ) and local (fluctuation spectrum, $\varrho(k)$) cases, the final probability density is independent of both the details of the partial wave construction and Λ (except as a limit of integration). It is also independent of the choice of the family of window functions $\mathcal{W}_\Lambda(k; k_H)$, and depends only on the limit $\lim_{\Lambda \rightarrow \infty} \mathcal{W}_\Lambda(k; k_H) = \mathcal{W}(k; k_H)$. Therefore the regulator can be removed by sending $\Lambda \rightarrow \infty$. Moreover, the boundary condition at $k = \Lambda$ becomes irrelevant in this limit, which is a familiar result from the theory of Sturm–Liouville operators. As a consistency

check, one can integrate $\mathbf{P}[\varrho(k)]$ with the condition $\int d \ln k \varrho(k) = \epsilon$ in order to obtain $\mathbf{P}(\epsilon)$.

We conclude by relating our result to earlier work by Ivanov [76] and Bullock & Primack [75], who studied the effect of non-Gaussianities from inflation on PBH formation, but obtained (naïvely) opposite results. In Ref. [76], it was found that non-Gaussianities reduce the probability for PBH formation, whereas in Ref. [75], using a different formalism, the formation probability was found to be enhanced. Consider the probability density for the total fluctuation, ϵ . The non-Gaussian correction conserves probability, so it can either act by moving probability from the tails into the central region, or by moving probability from the central region into the tails. (This effect was also observed in Refs. [68, 69].) In the former case, the probability of large fluctuations—and hence the fraction of the universe going into large collapsed objects—is diminished; in the latter case, this situation is reversed. Exactly which occurs depends on the sign of the coefficient J . When $J > 0$, the volume of the central region is increased at the expense of the tails. When $J < 0$, the volume of probability in the tails is increased at the expense of the central region. The crossover from central region to tails occurs at a threshold amplitude $\epsilon_* = \sqrt{3}\Sigma$. A priori, it would appear that J can have either sign, depending on the model of inflation under consideration. This goes some way towards reconciling the apparently divergent results of Refs. [75, 76].

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